# CHAPTER

## Why do we need a conic-section compass?

Henk Hietbrink[1], Hazal Yürük[2]

[1] independent scholar

E-mail: hietbrink.h@planet.nl

[2] Master student, Istanbul Technical University Department of Mathematics

E-mail: yuruk22@itu.edu.tr

**Abstract**

The Istanbul Museum of the History of Science and Technology in Islam displays so-called conic-section compasses or "perfect compasses". They are fascinating instruments that raise many questions, such as how they rotate on their axes and how the configuration of the two angles affect the drawing. An ordinary pair of compasses produces circles, the conic-section compass can draw all conic sections, even a straight line. Those who study its history go back more than a thousand years. The idea is straightforward. After a brief introduction, one would expect the instrument to fulfil its purpose. After trying out the instrument, one may wonder how the old scholars utilized it. The main drawback of the instrument is that it is not nearly as user-friendly as the ordinary compass. The clumsy part is that the end of the compass arm has to be extended or shortened to get the pencil to touch the paper. An important claim is that the instrument can solve mathematical equations. As an engineer and a mathematician, we address the following mathematical questions:

- Given the configuration angles, can we predict what the instrument will draw?
- Given the characteristics of the parabola or ellipse, can we compute the configuration to produce that specific conic section?

The answers determine whether the tool is primarily a didactic tool for teaching mathematics, or whether the tool is useful as a precision tool for solving mathematical equations graphically. Our aim is to provide an explanation of the functionality of the conic-section compass's to both historians and mathematicians. It is our goal to simplify the mathematical principles to people without a mathematical background. Basic high school algebra and ordinary trigonometry skills are sufficient to understand this text. We created a conic-section compass from leftover materials to demonstrate that any teacher can build a conic-section compass. It is our wish that teachers and pupils build one too.

All kinds of (practical) questions arise during the performance. By redoing historical experiments, the participants discover all kinds of details that are not mentioned in the original texts, that might be overlooked, but that turn out to be of decisive importance. In a conic-section compass, for example, fixing the angles between the drawing plane, the rotating axis and the telescopic arm is important. Mechanics play a role. Redoing, replicating, re-enactment, and reconstruction are performative methods which proved to have a positive impact in understanding history of science, in pedagogy and public outreach. In our opinion, reading and doing go hand in hand for true understanding. It can help to go from the library to the laboratory and back again.

**Keywords**:, al-Qūhī, al-Siğzī, al-Bīrūnī, conic-section compass, perfect compass, Euclidian compass, doubling a cube, gardeners construction, parabola, ellipse, performative methods

## 0. Introduction

The 17th-century Dutch mathematician Frans van Schooten (1615-1660) wrote a book on mechanical devices to trace hyperbolas, ellipses, and parabolas. Part of Hietbrink's website www.fransvanschooten.nl is devoted to his book "De Organica Conicarum Sectionum …". Each of his devices can draw only one specific conic section, for example, an ellipse. The conic-section compass is a much more universal instrument. It can draw all conic sections, even a straight line. For this reason, it is referred to as the perfect compass. Its roots go back to 10th-century Arabic sources, for instance, al-Qūhī, al-Siğzī, and al-Bīrūnī. One might expect conic-section compasses in the Greek/Roman era. Euclid mentions in book XI cutting cones at a right angle. The great geometer Apollonius mentioned all kinds of cones and sections, but there are no traces of a drawing device. Without physical evidence, we can only speculate. Maybe they had one, maybe not, maybe they didn't need one, because they had alternatives. We don't know, but the idea of an alternative is tempting. Later on in the European Renaissance, there was a revival of the conic-section compass. The quest for continuous drawing curves served one goal: to prove that continuous curves do exist.

Figure 1 shows examples of Arabic manuscript drawings of a conic-section compass.

**Figure 1**: Examples of conic-section compasses in Arabic manuscripts, from left to right, (a) al-Qūhī (b) al-Siğzī , (c) Abu l-Qasim al-Asturlabi

Figure 2 shows examples of the ellipse drawers of Frans van Schooten (17th century). Rose (1970) presented an overview of European Renaissance conic-section compasses while Raynaud (2007) discussed the connection between the Arabic mathematical tradition (10th to 12th century) and the European ones (13th to 15th century).

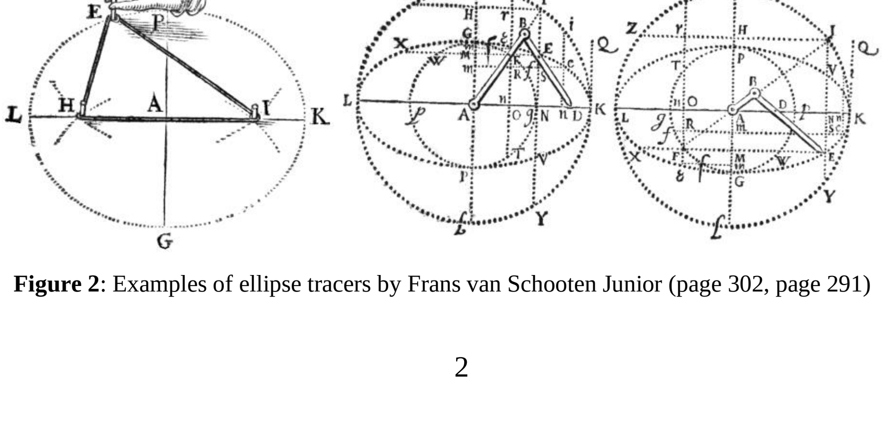

**Figure 2**: Examples of ellipse tracers by Frans van Schooten Junior (page 302, page 291)

In the 19th and 20th centuries, the question was raised as to whether the device could be used to graphically solve equations of the third degree or of the fourth degree. For the graphical solution, it is essential that the device draws accurately enough and that the preliminary mathematical calculations are feasible. The use of the conic-section compass may seem simple, but determining the appropriate angles for drawing a specific conic section requires tough initial calculations to be carried out. It will become evident that meticulous mathematical preparation is just as essential as skilful use of the instrument. In this contribution, we discuss the mathematics and proof the validity of the configuration formulas. We will show that without modern calculators or computers, the initial computational work to set up this device correctly is a serious obstacle. We are sceptical about the practicality of employing a universal conic-section compass. Technical innovations may improve the accuracy of the drawing, but cannot solve all mechanical problems. Moreover, in order to draw two parabolas and determine their intersection point, it is recommended to use two custom-made conic-section compasses instead of a single universal one. Using two dedicated compasses eliminates the need for initial mathematical calculations. Therefore, we believe that the conic-section compass is well suited for mathematical education, but inadequate for resolution of equations graphically.

The conic-section compass is a mathematical instrument. For a proper understanding, it is essential to understand its mathematics. Our aim is to provide an explanation of the functionality of the conic-section compass's to both historians and mathematicians. It is our goal to simplify the mathematical principles to people without a mathematical background. Basic high school algebra and ordinary trigonometry skills are sufficient to understand this text. We created a conic-section compass from leftover materials to demonstrate that any teacher can build a conic-section compass. It is our wish that teachers and pupils build one too. It illustrates the ingenuity of the ancient geometers who made tremendous progress without our modern computing devices.

In a previous contribution, we addressed the importance of performative methods (Hietbrink, 2021). Students of History of Science deserve to learn about these performative methods and do science themselves to find out on the conditions for failure or success. You won't get an idea of this experiment until you try to make a parabola or ellipse (Hendriksen, 2020).

Successively, the following sections address:

1. What does a conic-section compass do?
2. Does the conic-section compass deliver what it should do?
3. Where are the sources and manuscripts?
4. Can we solve the problem of doubling the cube?
5. Can we draw parabolas?
6. Can we draw ellipses?
7. Conclusions

**0. What is a conic section?**

For those who missed something, the history of the mathematics of conic sections began 2500 years ago. A conic section is the planar cut of a cone. Figure 3 shows four different cuts. On the left, the cut is a circle. The cut is perpendicular to the axis of the cone. The next cut is a parabola. Here, the cut is parallel to a side of the cone. The angle $\alpha$ between the axis and the cut equals half the topangle $\beta$. Euclid described cuts at a right angle. Therefore, the drawing shows a right angled cone, but according to Apollonius, it could have been any cone, but for a parabola always $\alpha = \beta$. The next intersection is an ellipse. The angle $\alpha > \beta$, the cut passes through the entire cone. At the right, the cut is a hyperbola. Now the angle $\alpha < \beta$, the cut doesn't go through the opposite side.

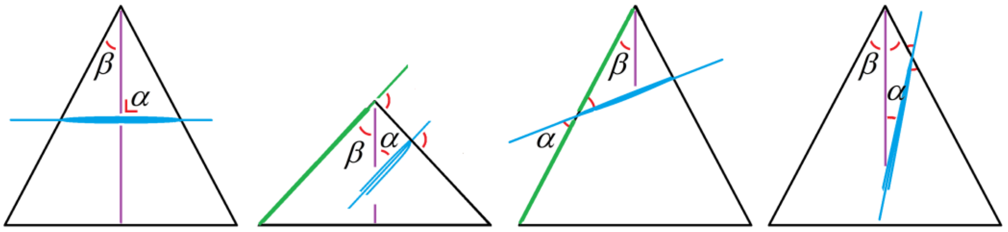


**Figure 3**: Four examples of conic sections: a circle, a parabola, an ellipse, and an hyperbola

### 1. What does a conic-section compass do?

The earliest descriptions of a conic-section compass are found in Arabic 10th century manuscripts. Replicas are exhibited in museums like the Istanbul Museum of the History of Science and Technology in Islam. In the 19th century, German and French scholars began to read Arabic mathematical texts, and Franz Woepcke was one of them. His drawing bears remarkable resemblance to what he discovered in his sources, for example the drawing of al-Qūhī. More pictures can be found in books printed in the European Renaissance (Raynaud 2007).

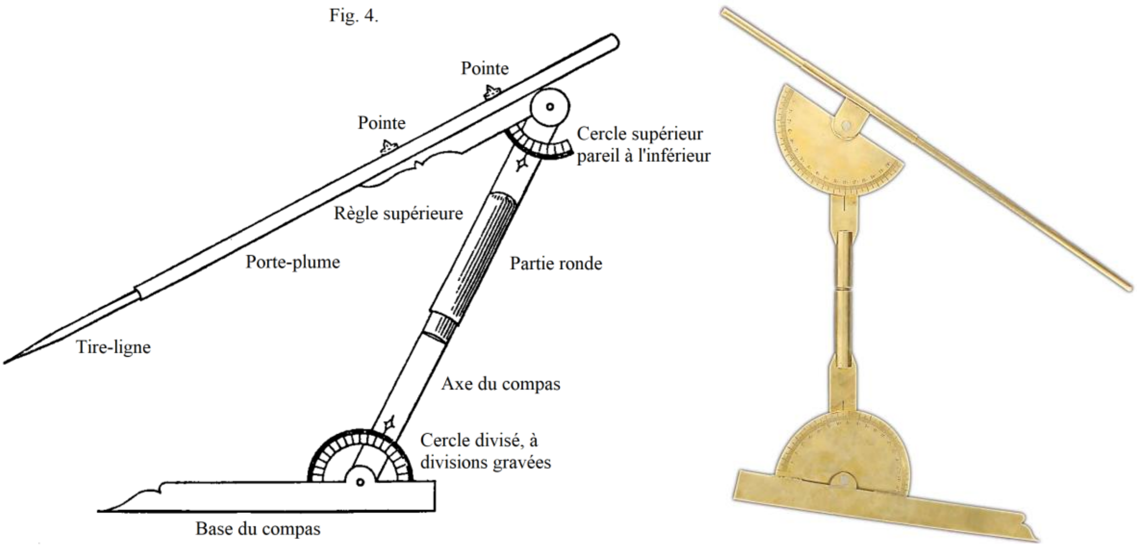


**Figure 4**: On the left Woepcke's drawing, on the right one of the IMHSTI museum's replicas.

Let's examine how the tool replicates the shape of a cone.

- Each cone has an axis. The compass fixed axis serves as the axis of the cone.
- A cone has a top angle. The angle between the fixed axis and the telescopic arm is half of the top angle.
- The surface of a cone can be described by a bunch of lines, all through the top of the cone with half of the top angle with the axis. When we rotate the telescopic arm, it generates the surface of the cone in a natural way: the telescopic arm is like a line in the cone surface.
- Each cone can be cut by a plane. The instrument is mounted to the drawing plane.

- The axis and the arm are tightly clamped to ensure that while turning, the angles do not change. Therefore, the rotation of the axis is as controlled as the rotation of an ordinary compass.
- Purpose of the telescopic arm is to bring the pencil to the drawing plane. While rotating the arm, the telescopic mechanism extends or retracts to bring the pen to the drawing plane.

Due to the nature of the surface of the cone, the length of the telescopic arm needs continuous adjustment. In mathematical terms, all we want is to mimic the process of prolonging a line. So, for a mathematician, the problem of bringing the pencil to the paper is not his problem, but a minor detail to be solved by an engineer.

**2. Does the conic-section compass deliver what it should do?**

A favourite text is by al-Qūhī. He starts his text with the possibility that the instrument can do nothing at all, no line, no circle, nothing. Then he proves that the instrument can produce all conic sections. In practice, it generates curves that look like lines, circles, ellipses, parabola, and hyperbola. So, in theory, the conic-section compass keeps its promise, but it has proven difficult to produce a reliable compass that convincingly produces conic sections.

**2.1. Which conic sections does a conic-section compass draw**

There are two issues to consider: determining the type of conic section produced and obtaining the mathematically exact measurements of the section produced. An important feature of the conic-section compass are the adjustable angles $\alpha$ and $\beta$. Figure 5 shows a relative small angle $\alpha$ between the drawing plane and the axis of the cone. Angle $\beta$ stands for half of the cone top angle. Together they determine the shape of the conic section. In case:

- $\alpha = 90^\circ$ and $\beta = 90^\circ$, then the axis is perpendicular to the drawing plane and the telescopic arm is parallel to the drawing plane, so the compass draws nothing
- $\alpha = 90^\circ$ and $\beta < 90^\circ$, then the axis is perpendicular to the drawing plane and the telescopic arm draws a circle
- $\alpha \neq 90^\circ$ and $\beta = 90^\circ$, then the compass draws a straight line
- $\alpha > \beta$, then the compass draws an ellipse
- $\alpha = \beta \neq 90^\circ$, then the compass draws a parabola
- $\alpha < \beta$, then the compass draws one branch of an hyperbola

In the example in Figure 5, because $\alpha < \beta$, the compass generates a branch of an hyperbola. Notice that the pen is pointing up. The compass can only draw a branch on the left side when the pen is pointing downwards, but it cannot draw a branch on the right side when the pen is pointing upwards. See figure 3 for more examples.

It may be surprising that the instrument is capable of not drawing at all. To resolve this, mount the fixed axis perpendicular to the drawing plane and align the telescopic arm parallel to the plane. This creates a situation with two right angles. Now, the rotation of the telescopic arm remains on a plane parallel to the drawing plane, ensuring that the pen tip cannot reach the drawing surface.

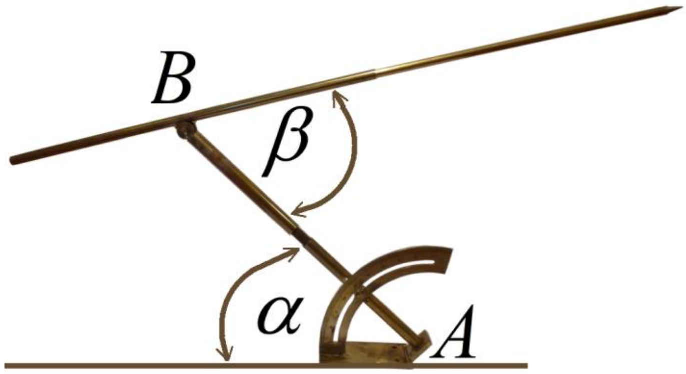


**Figure 5**: Another IMHSTI museum replica showing the two angles determine the shape of the conic section

Surprisingly, the instrument can draw a straight line. To achieve this result, position the fixed axis in any orientation (excluding perpendicular to the drawing plane) and fix the telescopic arm at a right angle. This results in the rotation of the telescopic arm in a plane. Now the cone is no longer a true cone with a single top because the angle between two sides totals 180°. So, we have a plane without a top. This plane intersects the drawing plane with a straight line. That is why the conic-section compass is able to draw a straight line. However, when attempting to draw a line in this manner, it may be difficult to adjust the length of the telescopic arm without disrupting the angles, possibly causing the line to appear as a circular section or a jagged line.

### 2.2 Conic section measurements

Second issue is how to configure the conic-section compass when one has a specific conic section in mind. Finding the proper configuration means determining the angles of the compass. In a next paragraph we deal with the configuration of the parabola. After that, we will provide the configuration of the ellipse. The conic-section compass is a mathematical instrument. For a proper understanding, it is essential to explain the relationship between the technical specifications of the instrument and the mathematical properties of circles, ellipses, and parabolas. In anticipation of the details, it is important to note that the mounting spot of the fixed axis, which is the axis of the cone, is definitely not the focal point of the conic section.

### 2.3. Practical considerations

Re-enacting is key in understanding historical instruments. The experience of reconstructing physically an instrument is not only of pedagogical value, but also provides insights that are not immediately apparent to one reading the texts. Bringing knowledge from the library to the laboratory and back again (Fors, 2016) sheds light on details that might be overlooked. For example, the telescoping mechanism proved to be a problem in practice because a telescopic arm is made up of several tubes that slide over each other. Sliding the tubes apart makes the arm longer, sliding the tubes together makes the arm shorter. The tighter the tubes are together, the better they line up, but the more resistance there is. The looser they are together, the easier they move. But the looser they are, the less in line they are. The greater the resistance, the more force is required to pull the tubes apart and the greater the risk of bending the set angle. Imperfection of the movement due to friction is inevitable! Friction prevents smooth operation. Adjusting the telescopic arm manually results in a pointwise approach rather than the ideal smooth curve. Moreover, it is quite difficult to manually adjust the telescope mechanism without disturbing the angles. During our experiments we were able to produce a curve, but after each round the curve was slightly different, not much, but too much to ignore. The difference is much more than the width of a pencil mark.

There are museum instruments with all kinds of stabilizers to prevent meandering, but stabilizers don't solve the friction problem of drawing consecutive points instead of a continuous curve. Therefore, the conic-section compass is a useful tool in a classroom when used by a skilled teacher with an enthusiastic group of students. For the same reason, the Van Schooten instruments have problems to draw proper curves. Due to the mechanical friction, producing a smooth parabolic shape was challenging. While animating in a digital environment shows the intended result, real attempts fail to produce desired results.

**3. Sources and manuscripts**

Sources on the conic-section compass are available. The primary sources are in Arabic, while the secondary sources are mainly in English, French, Italian or German. Arabic writing scholars have written advanced treatises about the conic-section compass. They called it the perfect compass. Given the length of the fixed, mounted axis, they did a lot of advanced geometry to set up the device correctly. Manuscripts on this compass have been studied by 19th and 20th century scholars. Critical editions on the Arabic text have been provided (Woepcke 1874), (Rashed, 2003, 2005). The outline of the text of al-Qūhī, made available by Woepcke and Rashed clearly shows their mathematical nature. Only a mathematician would begin a book by stating that a device cannot draw a curve and continue by stating that the device can draw a straight line. It is then shown that the device can draw a circle. To a mathematician, this is of great significance. However, one with less mathematical attitude would argue that by keeping the cap on the pencil there is neither a curve; instead, a ruler should be used to draw a line, and a conventional compass should be used to draw a circle. After this introduction, al-Qūhī discussed how his device can draw a hyperbola, parabola or ellipse.

Franz Woepcke died prematurely. Jules Mohl completed his work on the compass parfait (in French). To begin, Woepcke conducts an analytical verification utilizing Dandelin spheres. He concludes with formulas that resemble the solution of a quadratic equation. Woepcke transcribed the manuscripts of al-Husayn (12th century), al-Qūhī (10th century), and al-Sijzī (10th century). The primary objective of these manuscripts is to demonstrate that the conic-section compass does indeed produce conic sections.

A century later, Rashed (2005) presented transcriptions of additional Arabic manuscripts. He also included formulas resembling solutions of quadratic equations. Rashed made these manuscripts of al-Qūhī (10th century), al-Sijzī (10th century), al-Bīrūnī (10th century) available for scholars who cannot read Arabic. Rashed’s analysis of al-Qūhī's manuscript mentions the method for determining the angles required for a precise compass that generates a specific ellipse or parabola. However, this account is incomplete, as a crucial part is contained in al-Qūhī's missing work. The manuscript of al-Sijzī also fails to address how to determine the angles. Luckily, there is good news too: al-Bīrūnī's manuscript provides a complete set of instructions for computing angles using the compass. His constructions are ingenious. To calculate the angles, one must measure the length of the compass axis, indicate the length of the ellipse's major diameter, and indicate the length of the latus rectum, the crucial parameter that indicates the shape of a conic section. Figure 6 gives an impression of his geometrical construction. Start with the input at $A$, $B$, $C$, etc.., keep drawing lines and semicircles, keep marking points and keep drawing more lines and finally measure the angles $\angle OSQ$ and $\angle QOS$. al-Bīrūnī admits that the construction is complicated, but emphasizes that the result with the perfect compass is much finer than a point-by-point approximation. We verified all constructions using the geometrical software GeoGebra. His geometric construction works. It does what it is supposed to do. The construction itself is impressive. We deeply respect al-Bīrūnī who discovered and wrote down this algorithm centuries ago without computer, internet, and AI.

Later on, we will explain that solving a quadratic equation is a crucial step in the configuration process. Woepcke couldn't do without and al-Bīrūnī solves them by drawing three semicircles. Each indicates the squaring of rectangles: $a^2 = b \cdot c$ . This sequence of squaring rectangles is the geometrical equivalence of the algebraic operation of taking a square root, an essential operation in solving a quadratic equation. The purpose of Figure 6 is to encourage students to investigate al-Bīrūnī's ancient marvel and turn the classical geometry into modern algebra. A mathematical explanation of the geometrical construction is available at the website [https://fransvanschooten.nl/perfectcompass.htm](https://fransvanschooten.nl/perfectcompass.htm).

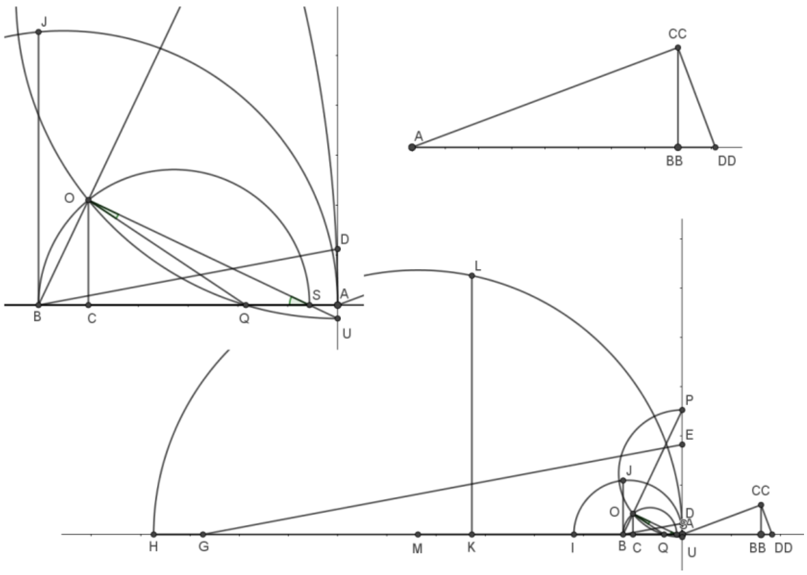


**Figure 6**: Geometrical construction of al-Bīrūnī

## 4. Can we solve the problem of doubling the cube?

One of the classical Greek geometry problems is that of doubling a cube. Given the edge of a cube, the problem requires the construction of the edge of a second cube whose volume is double that of the first. In 1837 Pierre Wantzel (1814–1848) proved that it is impossible to solve this problem by means of a classical geometrical construction using only a compass and a ruler. However, there were other tools. The conic-section compass is one of them.

Blåsjö (2021) wrote challenging contributions to the understanding of Greek geometry and their attempts to resolve the problem of doubling a cube. He suggested that there are good reasons to assume that the Greek geometers were using something like a conic-section compass. Blåsjö argues that in the Euclidean tradition, the Greek geometers were in need of two distinct right-angled conic-section compasses for two well-defined parabolas, to be precise: $y = x^2$ and $y^2 = 2x$ .

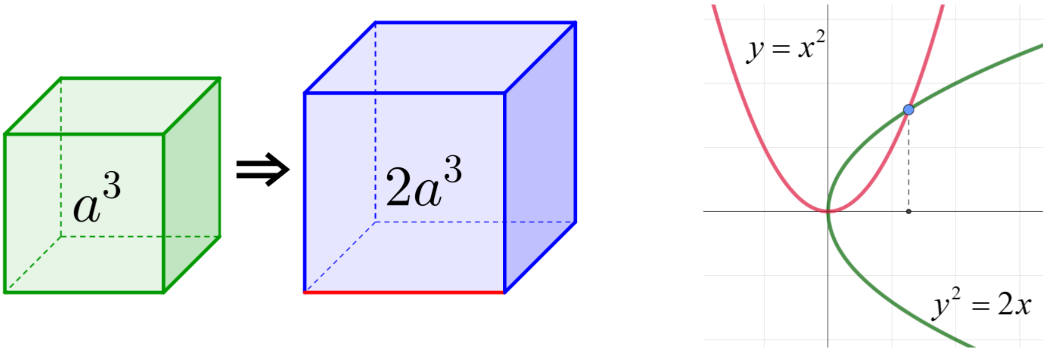


**Figure 7**: On the left a cube and a double volume cube, on the right the solution

Graphically you can plot the intersection of these two parabolas. In the intersection point of the parabolas $y = x^2$ and $y^2 = 2x$ holds the equation $\left(x^2\right)^2 = 2x$. Simplifying this equation results in $x^3 = 2$. The solution is $x = \sqrt[3]{2}$. This value represents the factor to enlarge a side for doubling a cube. Figure 7 shows a green cube with sides $a$ and a blue cube with sides $x \cdot a$ whose volume is double that of the green one.

### 5 Can we draw parabolas?

In order to solve the problem of doubling the cube graphically, we need two parabolas. In this section we show that the conic-section compass can do. Determination of the angles of a conic-section compass is part of the configuration process. The angles determine the shape of the conic section. See figure 3 for examples. First we explain the configuration and the working procedure of the Euclidean parabola compass, which is a dedicated conic-section compass. Result is a compass to draw the parabola $y^2 = 2x$. Next is the configuration of a universal conic-section compass to draw the second parabola $y = x^2$. We proof that there is a general configuration formula for drawing any $k \cdot y = x^2$. Next we discuss the alternative, another specific Euclidean parabola compass to draw $y = x^2$. Finally we discuss which is best: two dedicated conic-section compasses or a single universal one.

### 5.0 Euclidean parabola compass

Euclid's propositions deal with cuts perpendicular to the cone surface of a cone. If the cone has a right angle at its apex, the outline of the cut has the shape of a parabola with the length $AB$ of the axis as the parameter. In Figure 8, the axis $AB$ of a right-angled conic-section compass makes a $\alpha = \beta = 45°$ angle with the drawing plane all the time. Point $P_0$ is the top of the parabola. Line $AP_0$ is the line of symmetry of the parabola. In point $P_0$, this line has a right angle with the surface of the cone. By rotating the pencil around the instrument axis, a telescoping mechanism extends the leg $AP$ with the pencil. We started in position $P_0$. After half a rotation, a 180° turn, the leg $AP$ aligns parallel to the drawing plane. In this orientation, it cannot make contact with the drawing plane. One may say that point $P$ is at infinity. As it continues to rotate around the axis of the instrument, the telescopic arm retracts the leg $AP$ and completes the other half of the parabola.

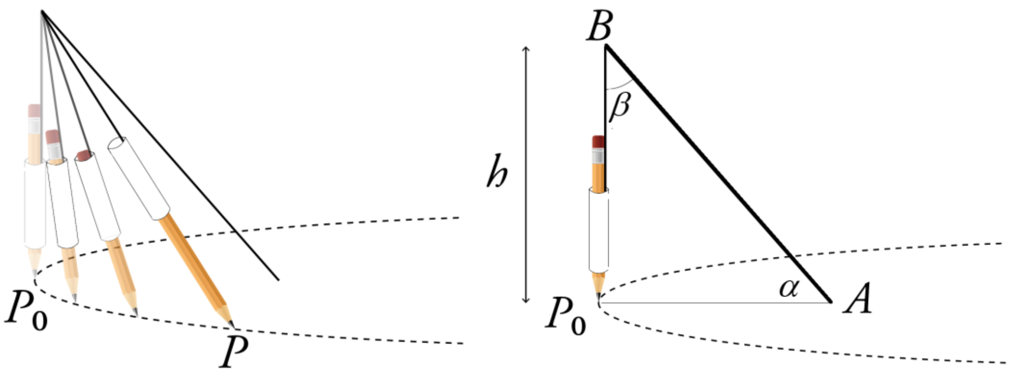


**Figure 8**: The construction of a parabola (Blåsjö)

### 5.1 The construction of the first parabola

Say, for example $AP_0 = 1$, so $AB = \sqrt{2}$, because $\alpha = \beta = 45°$. Say $P_0 = (0,0)$, so $A = (1,0)$. Figure 9 displays the conic-section compass laid flat on the drawing plane to provide clear visualization of relevant triangles. On the left, $P_0$ has been drawn to mark the top of the parabola. The axis of the conic-section compass is the hypotenuse of the triangle. Its length is $\sqrt{2}$. The length of the leg with the pencil is $1$. On the right, a point $B$ on the line perpendicular to the line of symmetry has been drawn. The axis of the conic-section compass is now one of the legs of the rectangular triangle. The side with the pencil has become the hypotenuse. Its length is $2$.

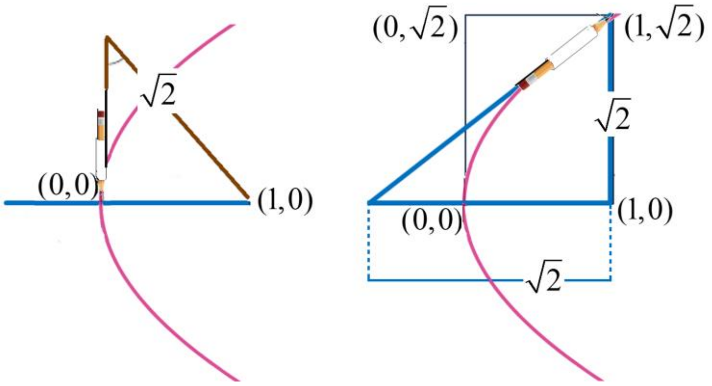


**Figure 9**: Specifying the parabola

Now we have two points on the parabola, point $P_0 = (0,0)$ and a point $B = (1,\sqrt{2})$ and the corresponding parabola is $2x = y^2$ with focal point $F = (\frac{1}{2},0)$. In general, a right angled conic-section compass with length $AB = \frac{1}{2}\sqrt{2}\cdot k$ (so $AP_0 = \frac{1}{2}\cdot k$), produces a parabola $k \cdot x = y^2$ with the corresponding points $P_0 = (0,0)$, $A = (\frac{1}{2}k,0)$, $B = (\frac{1}{2}k,\frac{1}{2}\sqrt{2}k)$, and focal point $F = (\frac{1}{4}k,0)$.

### 5.2 The construction of the second parabola

So far, we paid attention to the parabola $2x = y^2$. Next is the parabola $y = x^2$. Now, we have two options. Either we demand for another conic-section compass with $k = 1$ and $AB = \frac{1}{2}\sqrt{2}$, half the size of the first conic-section compass or we try to adjust the angles of the conic-section compass we already have. When the computational effort is small, we can choose the latter option, but when the mathematical effort gets tough, we might prefer the former option and build a second right-angled conic-section compass.

When we want to use the universal conic-section compass we already have, the one with $AB=\sqrt{2}$, we have to figure out which angle $\alpha$ produces the parabola $y=x^2$. It won't be a right-angled cone but an acute cone. The cutting is no longer at a right angle, but the same angle $\alpha$. Later on, we will proof the expression $\cos(\alpha)=\frac{\sqrt{66}-\sqrt{2}}{8}$, so the angle at the mounting point (also half the angle of the top of the cone) is $\alpha\approx 33°$. In general, in order to draw parabola $k\cdot y=x^2$ with a conic-section compass with length of the fixed axis $AB=l$, we can find angle $\alpha$ from expression $\cos\alpha=\frac{\sqrt{\frac{1}{16}k^2+l^2}-\frac{1}{4}k}{l}$. Before we prove this formula, we explain that a geometer can construct this angle very well. We skipped the explanation of al-Bīrūnī's construction because it was too long~, but this one is less complicated. Given $l$ and $k$, any geometer can construct this angle in the manner of Figure 10.

The construction starts with given $l$ and $k$. Draw a rectangle with points $O$, $K$, and $V$ where $OV=l$ and $KO=k$. Draw a circle with radius $l$ around point $V$, draw a circle with radius $\frac{1}{4}k$ around point $O$, marking point $N$, draw another circle with radius $\frac{1}{4}k$ around point $N$, marking point $P$. Finally, draw a line tangent to the circle through point $P$ and mark intersection point $Q$. Angle $\angle NVQ$ is what we were looking for: angle $\alpha$.

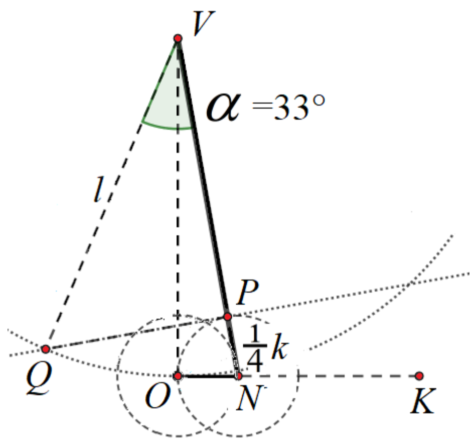


**Figure 10**: A geometrical construction of angle

Now we can say that once we have the configuration formula, both the geometrical construction and the arithmetic computation of the angle for the conic-section compass are feasible. One can do without a graphing calculator or a computing device.

| **Table 1.** Summary of formulas for a parabola | |
|---|---|
| Property of the width of a parabola: latus rectum $k$. | $k\cdot y=x^2$ |
| Ratio of $AP_0$, the distance between the mounting point $A$ to the top of the parabola $P_0$, and the length of the fixed axis $AB$. | $\frac{AP_0}{AB}=\frac{1}{2\cos\alpha}$ |
| Ratio of $k$, the length of the latus rectum, to the length of the fixed axis $AB$. | $\frac{k}{AB}=2\cdot\tan\alpha\cdot\sin\alpha$ |
| Computation of the angle $\alpha$. | $\cos\alpha=\frac{\sqrt{\frac{1}{16}k^2+AB^2}-\frac{1}{4}k}{AB}$ |

### 5.3 Proof of the formula

Next we proof this formula $\cos\alpha = \frac{\sqrt{\frac{1}{16}k^2 + l^2} - \frac{1}{4}k}{l}$. Figure 11 shows parabola $k \cdot y = x^2$, point $P_0 = (0,0)$, and $AB = l$. Say $A = (a,0)$ and $D = (a,b)$, where $k \cdot a = b^2$. It is about time to explain the importance of $k$, the so called latus rectum. Its meaning is the length of the chord perpendicular to the parabola's line of symmetry that passes through the focal point $F$. The focal point is irrelevant to understanding the operation of the conic-section compass, but we mention it because the concept is frequently associated with conic sections.

Angle $\alpha$ is present in triangle $ABP_0$ in points $A$ and $B$, and also in rectangular triangle $ACD$ in point $C$. According to the sine rule holds $\frac{l}{\sin(180° - 2\alpha)} = \frac{a}{\sin(\alpha)}$. Because $\sin(180° - 2\alpha) = \sin(2\alpha)$ and $\sin(2\alpha) = 2 \cdot \sin(\alpha) \cdot \cos(\alpha)$, so $a = \frac{\frac{1}{2} \cdot l}{\cos\alpha}$. In the rectangular triangle holds $\frac{b}{l} = \tan\alpha$. Because $k \cdot a = b^2$ we obtain equation $k \cdot \frac{\frac{1}{2} \cdot l}{\cos\alpha} = (l \cdot \tan\alpha)^2$ thus $k \cdot \cos\alpha = 2 \cdot l \cdot (1 - \cos^2\alpha)$ thus $2 \cdot l \cdot \cos^2\alpha + k \cdot \cos\alpha - 2 \cdot l = 0$ and therefore $\cos\alpha = \frac{-k + \sqrt{k^2 + 16 \cdot l^2}}{4 \cdot l}$ or simplified $\cos\alpha = \frac{\sqrt{\frac{1}{16}k^2 + l^2} - \frac{1}{4}k}{l}$ and thus $a = \frac{\frac{1}{2} \cdot l}{\cos\alpha}$ and $b = \sqrt{k \cdot a}$. As a result, these formulas demonstrate how to configure a conic-section compass for a specific parabola.

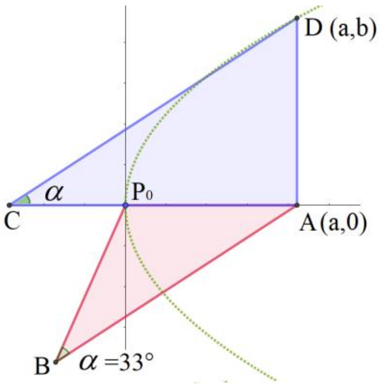


**Figure 11**: Construction of a parabola

These `configuration formulas help to construct any parabola (given $AB = l$ and $k$, find $\alpha$) or to understand which parabola the conic-section compass is going to draw (given $\alpha$ and $AB = l$, find $k$).

Given $AB = l$ and $k$, we can compute $\cos\alpha = \frac{\sqrt{\frac{1}{16}k^2 + l^2} - \frac{1}{4}k}{l}$.

Given $\alpha$ and $AB = l$, we can compute $k = 2 \cdot l \cdot \tan\alpha \cdot \sin\alpha$.

One final remark on the focal point $F$. This point is halfway between points $A$ and $P_0$ in case of the Euclidean parabola compass. A challenging student exercise is to find its position in the universal conic-section compass. Further on, it will become clear that it is definitely not halfway there.

### 5.4 Dedicated or Universal

Finally we discuss what is best: two dedicated right angled compasses or one universal conic-section compass. We constructed two parabolas $2x = y^2$ and $y = x^2$ with two dedicated conic-section compasses and marked the intersection point. We showed that it is also possible to do the job with a single universal conic-section compass. Among mathematicians the latter approach is more advanced than the former. Among non-mathematicians, we prefer the former to avoid the need of explaining what we are doing with the configuration formula to find the required angle. From a technical point of view, it is more easy to construct a stable right-angled conic-section compass than a flexible universal compass. In both ways, it needs craftsmanship to work with the device to produce the required curves. We did experiments with a home-made instrument, but because of the inaccuracy of our instrument and because of our inexperience to operate it, the value of $x = \sqrt[3]{2}$ could lie between $0{,}5$ and $2{,}5$, which is pretty bad. As an engineer, it was like a failure, but as mathematicians we proved that we can construct the cube root of two.

## 6. Can we draw ellipses?

Up to this point, we have drawn parabolas. Now, we will explore how to draw ellipses. A conic-section compass has an axis with fixed length $AB$ and two parameters: angle $\alpha$ and angle $\beta$. Angle $\alpha$ is the angle between the drawing plane and the fixed axis. Angle $\beta$ is the angle between the fixed axis and the telescopic arm. The shortest length of the arm is $BD$ and its longest length is $BE$. Angle $2\beta$ is the top angle of the cone. Figure 12 offers a sketch. The mathematical explanation of the construction of the ellipse requires more work than the one of the parabola. On one hand, this text aims to describe the resulting conic section that arises from the setup of a conic-section compass. On the other hand, the text provides instruction on how to set up the compass to achieve a desired conic section.

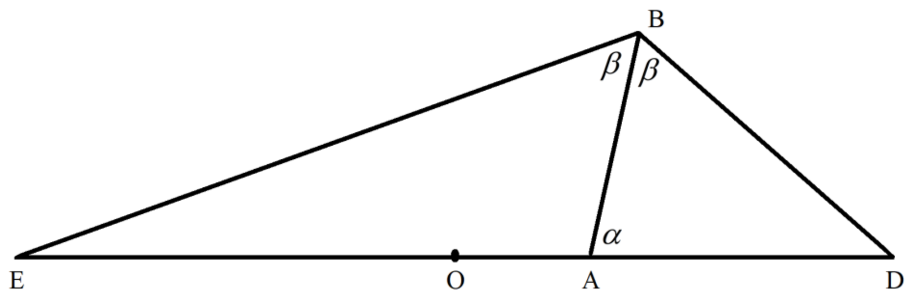


**Figure 12**: Sketch of a conic-section compass with axis $AB$, long diameter $DE$ and center $O$.

### 6.0 Roadmap

Figure 13 shows the properties of the ellipse $DHET$ with center $O$, long diameter $DE$, short diameter $HT$ and latus rectum $NQ$. The conic-section compass is shown three times in triangles $ABD$, $ABE$, and $AUG$. Note that $AB = AU$ and that $BD < GU < BE$. Table 2 summarizes the interconnected equations. The left hand represents the lengths $AB$, etc. , the right hand represents the trigonometrical expressions related to the angles $\alpha$ and $\beta$.

Let's start from the measurements of the conic-section compass, the angles $\alpha$ and $\beta$, and length $AB$, Table 2 shows that it is possible to calculate all lengths of the ellipse (or other conic section), like long diameter, short diameter, focal distance, eccentricity, etc. This is straight forward because $AB$, $\alpha$, and $\beta$ are input to the formulas in table 2. These are the steps. The length of the long diameter in an ellipse can be expressed in terms of $AB$, $\alpha$, and $\beta$ by applying the sine rule. This allows for the computation of the position of the ellipse's center $O$. Further algebraic steps are necessary to express the length of the short diameter $HT$ in terms of $AB$, $\alpha$, and $\beta$. More high school algebra is needed to develop a formula for both the eccentricity $\varepsilon$, the focal distance $RS$, and the latus rectum $NQ$.

Inversely, the route from conic section to conic-section compass, which means given all lengths including $AB$, we have to calculate angles $\alpha$ and $\beta$. Now we have to solve a set of equations, which is not a straight forward computation, but real algebraic solving a set of equations. Woepcke found a solution using focal points and Dandelin spheres. Rashed argues that the Arabic writing scholars didn't do it that way. We preferred a Euclidian approach with triangles and ratios between the length of the long and the short diameter. This way, when we write sine or cosine, the ancient geometers al-Qūhī, al-Siğzī, and al-Bīrūnī would have mentioned ratios of segments, praising the eminent geometer Apollonius for his propositions. Table 3 summarizes the outcomes of the angle calculation formulas. These formulas require the eccentricity $\varepsilon$ and the relative length of the latus rectum $w$. The mathematical deduction will be presented in a next section.

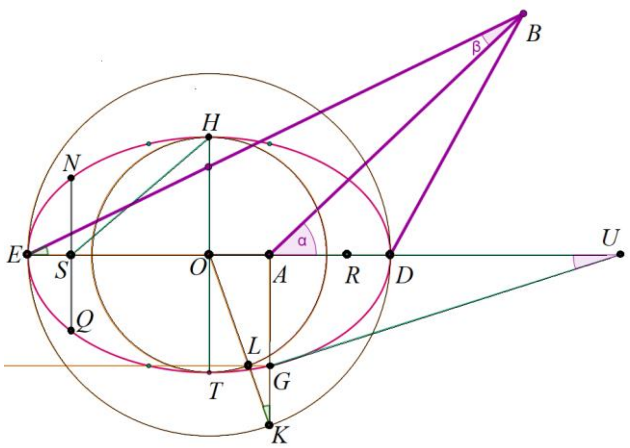


**Figure 13**: Sketch with lettering of a conic-section compass producing an ellipse.

**Table 2**. Overview of formulas for an ellipse

| | | |
|---|---|---|
| Ratio of $DO$, half of the length of the long diameter, to the length of the fixed axis $AB$. | $\frac{DO}{AB}=\frac{\sin\alpha\cdot\sin\beta\cdot\cos\beta}{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}$ | (1) |
| Relative position of the mounting point $AO$ to half of the length of the long diameter $DO$. | $\frac{AO}{DO}=\frac{\tan\beta}{\tan\alpha}$ | (2) |
| Ratio of $HO$, half of the length of the short diameter, to the length of the fixed axis $AB$. | $\frac{HO}{AB}=\frac{\sin\alpha\cdot\sin\beta}{\sqrt{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}}$ | (3) |
| Ratio of $DE$, the length of the long diameter, to the length of the short diameter $HT$. | $\frac{DE}{HT}=\frac{DO}{HO}=\frac{\cos\beta}{\sqrt{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}}$ | (4) |
| Eccentricity $\varepsilon$ is the ratio of $RS$, the length of the focal distance to the length of the long diameter $DE$. | $\varepsilon=\frac{RS}{DE}=\sqrt{1-\left(\frac{HO}{DO}\right)^2}=\frac{\cos\alpha}{\cos\beta}$ | (5) |
| Ratio of $NQ$, the length of the latus rectum, to the length of the fixed axis $AB$. | $\frac{k}{AB}=\frac{NQ}{AB}=2\cdot\frac{HO^2}{AB\cdot DO}=2\cdot\sin\alpha\cdot\tan\beta=w$ | (6) |

| **Table 3.** Overview of angle formulas for an ellipse | | |
|---|---|---|
| $\cos^2\alpha = \dfrac{(1+\varepsilon^2+\frac{1}{4}\omega^2)-\sqrt{(1+\varepsilon^2+\frac{1}{4}\omega^2)^2-4\varepsilon^2}}{2}$ | $0<\alpha<90°$ | (7) |
| $\cos^2\beta = \dfrac{(1+\varepsilon^2+\frac{1}{4}\omega^2)-\sqrt{(1+\varepsilon^2+\frac{1}{4}\omega^2)^2-4\varepsilon^2}}{2\varepsilon^2}$ | $0<\beta<90°$ | (8) |

The measurements of the ellipse can be expressed in the properties of the instrument:

- Length of axis $l = AB$
- Half of the top angle $\beta$
- Cutting angle $\alpha$

These three determine the measurements of the ellipse. The other way round, the angles of the conic-section compass can be expressed in terms of the length of the instrument and the measures of the ellipse.

- the length of the long diameter $a = DE = 2\cdot EO = 2\cdot DO$
- the length of the short diameter $b = HT = 2\cdot LO$
- the focal distance $c = RS = 2\cdot OR$ (note that $a^2 = b^2 + c^2$)
- the length of the latus rectum $k = NQ$

Two out of four determine the configuration of the conic-section compass.

### 6.1. Long diameter

The relationship between the length of axis $l = AB$ and long diameter $a = DE$ can be established by applying the sine rule in the triangles $ABD$, $ABE$ and $DBE$:

- $\dfrac{AD}{AB} = \dfrac{\sin\beta}{\sin(\alpha+\beta)}$
- $\dfrac{BD}{AB} = \dfrac{\sin\alpha}{\sin(\alpha+\beta)}$
- $\dfrac{BE}{AB} = \dfrac{\sin\alpha}{\sin(\alpha-\beta)}$
- $\dfrac{AE}{AB} = \dfrac{\sin\beta}{\sin(\alpha-\beta)}$

Furthermore, taking into account that $DE = AE + AD$:

- $\dfrac{DE}{AB} = \dfrac{AD+AE}{AB} = \dfrac{\sin\beta}{\sin(\alpha+\beta)} + \dfrac{\sin\beta}{\sin(\alpha-\beta)} = 2\cdot\dfrac{\sin\alpha\cdot\sin\beta\cdot\cos\beta}{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}$
- $\dfrac{DO}{AB} = \dfrac{EO}{AB} = \frac{1}{2}\dfrac{DE}{AB} = \dfrac{\sin\alpha\cdot\sin\beta\cdot\cos\beta}{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}$ (1)

- $\frac{AO}{AB}=\frac{DO-AD}{AB}=\frac{\sin\beta\cdot\sin\beta\cdot\cos\alpha}{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}$

- $\frac{BD}{AD}=\frac{BE}{AE}=\frac{BD+BE}{AD+AE}=\frac{BD+BE}{DE}=\frac{\sin\alpha}{\sin\beta}$ or $\frac{BD}{AD}=\frac{BD/AB}{AD/AB}=\frac{\frac{\sin\alpha}{\sin(\alpha+\beta)}}{\frac{\sin\beta}{\sin(\alpha+\beta)}}=\frac{\sin\alpha}{\sin\beta}$

### 6.2. Relative position of the mounting spot

The mounting spot is the point where the axis of the instrument is attached to the drawing plane. Due to symmetry, it must be somewhere on the long axis. The drawing shows that the mounting spot is not necessarily the same as the focal point. The relative position of the focal point will be discussed later. The ratio of the distance $AO$ between the center of the ellipse $O$ and the mounting point $A$, to half the length of the long diameter $DO$, turns out to be nice.

- $\frac{AO}{DO}=\frac{AO/AB}{DO/AB}=\frac{\frac{\sin\beta\cdot\sin\beta\cdot\cos\alpha}{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}}{\frac{\sin\alpha\cdot\sin\beta\cdot\cos\beta}{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}}=\frac{\sin\beta\cdot\sin\beta\cdot\cos\alpha}{\sin\alpha\cdot\sin\beta\cdot\cos\beta}=\frac{\sin\beta\cdot\cos\alpha}{\sin\alpha\cdot\cos\beta}=\frac{\tan\beta}{\tan\alpha}$ (2)

### 6.3. Short diameter

The relationship between the length of the axis $AB$ and the length of the short diameter $b=HT$ can be established by applying proposition 13, book four of Euclid's Elements: the construction of a square root. Say point $K$ is at the circle with diameter $DE$. Given $\alpha$, $\beta$, $AB$, and computed $AD$, and $AE$, let's find the length of segment $AK$:

- $AK=\sqrt{AD\cdot AE}$

The semicircles of al-Bīrūnī have been mentioned before. They are part of the algorithm of squaring a rectangle. The above relationship is an example: the rectangle has sides $AD$ and $AE$, and the square has sides $AK$. Now the area of the square is equal to the area of the rectangle $AK\cdot AK=AD\cdot DE$.

Like we did for the parabola, when we laid down the conic-section along the long diameter, the tip of the pen is at a line perpendicular to the long diameter and thus parallel to the short diameter. Now, point $U$ is the top of the instrument and point $G$ is the tip of the pen. Triangle $AUG$ has a right angle in point $A$ and side $AU=AB$. Therefore:

- $\frac{AG}{AB}=\tan\beta$

Say point $K$ is the intersection of the circle with long diameter $DE$ and line $AG$. Say point $L$ is the intersection of the circle with short diameter $HT$ and line $KO$.

- $\frac{1}{2}HT=HO=TO=LO$

- $\frac{1}{2}DE=DO=EO=KO$

- $\frac{LO}{KO}=\frac{AG}{AK}$

So, all together

- $\frac{HO}{AB}=\frac{LO}{AB}=\frac{AG}{AK}\cdot\frac{KO}{AB}=\frac{AB}{AB}\cdot\frac{AG}{AB}\cdot\frac{KO}{AK}=\frac{AB\cdot\tan\beta}{\sqrt{AD\cdot AE}}\cdot\frac{DO}{AB}=\frac{\tan\beta}{\sqrt{\frac{AD}{AB}\cdot\frac{AE}{AB}}}\cdot\frac{DO}{AB}=....$

$$....=\frac{\tan\beta}{\sqrt{\frac{\sin\beta}{\sin(\alpha+\beta)}\cdot\frac{\sin\beta}{\sin(\alpha-\beta)}}}\cdot\frac{\sin\alpha\cdot\sin\beta\cdot\cos\beta}{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}$$

Results in

- $\frac{HO}{AB}=\frac{\tan\beta}{\sqrt{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}}\cdot\frac{\sin\alpha\cdot\cos\beta}{1}=\frac{\sin\alpha\cdot\sin\beta}{\sqrt{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}}$ (3)

Finally, we have a relationship between the length of the small diameter $b$ and the length of the axis $AB$ because $b=HT=2\cdot HO=2\cdot LO$.

Another useful relation is the next one. Later on, we connect it to the latus rectum $k=NQ$.

- $\frac{DO}{HO}=\frac{DO}{LO}=\frac{\frac{DO}{AB}}{\frac{LO}{AB}}=\frac{\frac{\sin\alpha\cdot\sin\beta\cdot\cos\beta}{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}}{\frac{\sin\alpha\cdot\sin\beta}{\sqrt{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}}}=\frac{\cos\beta}{\sqrt{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}}$ (4)

- $\frac{LO^2}{AB\cdot DO}=\frac{LO}{AB}\cdot\frac{LO}{DO}=\frac{LO/AB}{DO/LO}=\frac{\frac{\sin\alpha\cdot\sin\beta}{\sqrt{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}}}{\frac{\cos\beta}{\sqrt{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}}}=\sin\alpha\cdot\tan\beta$

Angle $\angle AKO$ is of interest because its sine value is the same as the relative position of the mounting spot. See (2).

- $\sin\angle AKO=\frac{AO}{DO}=\frac{\tan\beta}{\tan\alpha}$

### 6.4. Eccentricity

Eccentricity is the ratio of the length of the focal distance to the length of the long diameter. Focal distance is $c=RS=2\cdot OR=\sqrt{a^2-b^2}$ because $OR^2+OH^2=OE^2$ like $b^2+c^2=a^2$. Note that $OH=OL$, so $OR^2=OE^2-OL^2$.

- $\varepsilon=\frac{c}{a}=\frac{OR}{EO}=\sqrt{1-\left(\frac{LO}{EO}\right)^2}=....$

$$....=\sqrt{1-\left(\frac{\frac{LO}{AB}}{\frac{EO}{AB}}\right)^2}=\sqrt{1-\left(\frac{\frac{\sin\alpha\cdot\sin\beta}{\sqrt{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}}}{\frac{\sin\alpha\cdot\sin\beta\cdot\cos\beta}{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}}\right)^2}=....$$

$$.... = \sqrt{1 - \left( \frac{\sqrt{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}}{\cos\beta} \right)^2} = ....$$

$$.... = \sqrt{1 - \frac{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}{\cos^2\beta}} = \sqrt{\frac{\cos^2\beta - \sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}{\cos^2\beta}} = ....$$

$$.... = \sqrt{\frac{\cos^2\beta - \left(\sin^2\alpha\cdot\cos^2\beta - \sin^2\beta\cdot\cos^2\alpha\right)}{\cos^2\beta}} = ....$$

$$.... = \sqrt{\frac{\cos^2\beta\cdot\left(1-\sin^2\alpha\right) + \sin^2\beta\cdot\cos^2\alpha}{\cos^2\beta}} = ....$$

$$.... = \sqrt{\frac{\cos^2\beta\cdot\cos^2\alpha + \sin^2\beta\cdot\cos^2\alpha}{\cos^2\beta}} = \sqrt{\frac{\left(\cos^2\beta + \sin^2\beta\right)\cdot\cos^2\alpha}{\cos^2\beta}} = \sqrt{\frac{\cos^2\alpha}{\cos^2\beta}}$$

Because $0 < \beta < \alpha < 90°$, the values of these cosines is always positive and there is no need for absolute values. Finally, the result for eccentricity is:

- $\varepsilon = \dfrac{\cos\alpha}{\cos\beta}$ (5)

### 6.5. Focal Distance

One more step is needed to express the length of the focal distance $RS$ in terms of $\alpha$, $\beta$, and $AB$:

- $RS = 2\cdot OR$
- $\varepsilon = \dfrac{RS}{DE} = \dfrac{OR}{EO}$
- $\dfrac{OR}{AB} = \dfrac{OR}{EO}\cdot\dfrac{EO}{AB} = \varepsilon\cdot\dfrac{EO}{AB}....$

$$.... = \frac{\cos\alpha}{\cos\beta}\cdot\frac{\sin\alpha\cdot\sin\beta\cdot\cos\beta}{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)} = ....$$

$$.... = \frac{\sin\alpha\cdot\sin\beta\cdot\cos\alpha}{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}$$

So the ratio of the length of the focal distance to the length of the fixed axis is:

- $\dfrac{RS}{AB} = 2\cdot\dfrac{\sin\alpha\cdot\sin\beta\cdot\cos\alpha}{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}$

### 6.6. Latus Rectum

Latus rectum is the last topic. There are two reasons for treating it. Firstly, al-Bīrūnī used latus rectum in his calculation scheme, because he was aware of the works of Apollonius who proved the so-called fundamental property of each type of conic section. In all three cases this property involves a segment that is called the latus rectum (Hogendijk, 1991). Secondly, we too need it to discover the reverse relationship: given the properties of the ellipse, discover the angles $\alpha$ and $\beta$ to generate that ellipse.

Latus rectum $k$ is the length of the chord parallel to the short diameter through the focal point. Its length is $NQ = k = \frac{b^2}{a} = a\left(1-\varepsilon^2\right)$ where $DE = a$ is the length of long diameter, where $HT = b$ is the length of short diameter, and where $\varepsilon$ is the eccentricity of the ellipse. Now we continue with $w$ what is defined as the relative length of the latus rectum.

- $$w = \frac{NQ}{AB} = \frac{k}{AB} = \frac{\frac{b^2}{a}}{AB} = \frac{HT/AB \, . \, HT/AB}{DE/AB} = 2 \cdot \frac{HO/AB \, . \, HO/AB}{DO/AB} = 2 \cdot \frac{HO^2}{AB \cdot DO}$$
$$... = 2 \cdot \frac{\frac{\sin\alpha \cdot \sin\beta}{\sqrt{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}} \cdot \frac{\sin\alpha \cdot \sin\beta}{\sqrt{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}}}{\frac{\sin\alpha\cdot\sin\beta\cdot\cos\beta}{\sin\left(\alpha+\beta\right)\cdot\sin\left(\alpha-\beta\right)}} = 2\cdot\sin\left(\alpha\right)\cdot\tan\left(\beta\right) \tag{6}$$

We have now reached the end of Table 2. All the ratios between lengths have been expressed in trigonometric formulas using angle $\alpha$ and $\beta$. Given these angles and given the length of the fixed axis $AB$, we can calculate all the measurements of the ellipse, such as long diameter, short diameter, focal distance, and latus rectum.

### 6.7. From Ellipse to Instrument

Up to this point, all relations started with given $\alpha$, $\beta$, and $AB$, which means, starting from the configuration of the conic-section compass. From now on, our aim is to reverse the relations in order to determine how to set up the device to create a desired ellipse. The long diameter and short diameter determine the full specification of an ellipse. Given ellipse properties of long diameter $DE = a$ and short diameter $HT = b$ and instrument property length $AB$, we are going to find expressions for the angles $\alpha$ and $\beta$ in terms of the length of the latus rectum $k$ and the eccentricity $\varepsilon$. We already mentioned $w$, the relative length of the latus rectum, the ratio of the length of latus rectum $k = NQ$ to the length of the fixed axis $l = AB$.

- $$\varepsilon = \frac{OR}{EO} = \frac{c}{a} = \frac{\cos\alpha}{\cos\beta} \qquad \text{(see 5)}$$

- $$w = \frac{k}{AB} = 2\cdot\sin\alpha\cdot\tan\beta \qquad \text{(see 6)}$$

For convenience, we introduce the intermediate variable $v = \left(\frac{w}{\varepsilon}\right)^2$, and express $v$ in terms of $\cos\beta$. In the end, we will express $\cos\beta$ in terms of $k$ and $\varepsilon$ without $v$. In case of an ellipse, always $k > 0$, $w > 0$, $0 < \varepsilon < 1$, $0 < \alpha < 90°$, and $0 < \beta < 90°$. The practical meaning of variable $v$ is less clear. Its purpose is to simplify the algebraic manipulations.

- $$\frac{w}{\varepsilon} = \frac{k/l}{c/a} = \frac{k\cdot a}{l\cdot c} = \frac{b^2/a \cdot a}{l\cdot c} = \frac{b^2}{l\cdot c}$$

- $$v = \left(\frac{w}{\varepsilon}\right)^2 = \left(\frac{2\cdot\sin\alpha\cdot\tan\beta}{\frac{\cos\alpha}{\cos\beta}}\right)^2 = \frac{4\cdot\sin^2\alpha\cdot\tan^2\beta}{\frac{\cos^2\alpha}{\cos^2\beta}} = \frac{4\cdot\sin^2\alpha\cdot\sin^2\beta}{\cos^2\alpha}$$

Let's start with the latter equation, aiming to find the formulas to calculate angles $\alpha$ and $\beta$. Lengths are known, so variables $\varepsilon$, $w$, and $v$ are known.

- Substitution of $\sin^2\alpha$ into $1-\cos^2\alpha$ results in $v\cdot\cos^2\alpha = 4\cdot\left(1-\cos^2\alpha\right)\cdot\sin^2\beta$
- Rearranging this into $\left(\frac{v}{4}+\sin^2\beta\right)\cdot\cos^2\alpha=\sin^2\beta$, and in $\cos^2\alpha=\frac{\sin^2\beta}{\frac{1}{4}v+\sin^2\beta}$
- Substitution of $\sin^2\beta$ into $1-\cos^2\beta$ results in $\cos^2\alpha=\frac{1-\cos^2\beta}{\frac{1}{4}v+1-\cos^2\beta}$.
- Because of (5) $\varepsilon=\frac{\cos\alpha}{\cos\beta}$, therefore $\varepsilon^2\cdot\cos^2\beta=\cos^2\alpha$
- so $\varepsilon^2\cdot\cos^2\beta=\frac{1-\cos^2\beta}{\frac{1}{4}v+1-\cos^2\beta}$, an expression with only $\cos\beta$ , $v$, and $\varepsilon$.
- Continue with $\varepsilon^2\cdot\cos^2\beta\cdot\left(\frac{1}{4}v+1-\cos^2\beta\right)=1-\cos^2\beta$ ,
- next $-\varepsilon^2\cdot\left(\cos^2\beta\right)^2+\left(\varepsilon^2\cdot\frac{1}{4}v+\varepsilon^2+1\right)\cdot\cos^2\beta-1=0$
- next $\varepsilon^2\cdot\left(\cos^2\beta\right)^2-\left(\varepsilon^2\cdot\frac{1}{4}v+\varepsilon^2+1\right)\cdot\cos^2\beta+1=0$
- because $w^2=v\cdot\varepsilon^2$ holds $\varepsilon^2\cdot\left(\cos^2\beta\right)^2-(1+\varepsilon^2+\frac{1}{4}w^2)\cdot\cos^2\beta+1=0$.

Now we have a quadratic equation for $\cos\beta$ , expressed in $w$ and $\varepsilon$.

- Solution is $\cos^2\beta=\frac{(1+\varepsilon^2+\frac{1}{4}w^2)-\sqrt{(1+\varepsilon^2+\frac{1}{4}w^2)^2-4\varepsilon^2}}{2\varepsilon^2}$ because $\cos^2\beta<1$. (7)
- Since $\varepsilon=\frac{\cos\alpha}{\cos\beta}$,
- therefore $\cos^2\alpha=\frac{(1+\varepsilon^2+\frac{1}{4}w^2)-\sqrt{(1+\varepsilon^2+\frac{1}{4}w^2)^2-4\varepsilon^2}}{2}$ because $\cos^2\alpha<1$. (8)

Finally we have expressions for $\cos\alpha$ and $\cos\beta$ that refer back to the ellipse properties $a$, $b$, $c$, $k$, and the length of the instrument $l$, in which:

- $a$ is the length of the long diameter $DE$
- $b$ is the length of the short diameter $HT$
- $c$ is the length of the focal distance $RS$
- $k$ is the length of the latus rectum $NQ$
- $l$ is the length of the instrument's fixed axis $AB$

Note that these formulas are only valid for ellipses. They are not valid for circles or parabolas. Table 4 shows some numerical examples of calculations for three different lengths of the fixed axis. The angles do not make nice round numbers, as you can clearly see.

**Table 4a**. Numerical examples for fixed axis $l = AB = 1$

| long diameter $a$ | short diameter $b$ | focal distance $c$ | eccentricity $\varepsilon$ | latus rectum $k$ | angle $\alpha$ | angle $\beta$ | relative position $AO/AB$ |
|---|---|---|---|---|---|---|---|
| 2 | 1 | 1,732 | 0,866 | 0,500 | 36,9° | 22,6° | 55,4% |
| 3 | 1 | 2,828 | 0,943 | 0,333 | 27,5° | 19,8° | 69,2% |
| 4 | 1 | 3,873 | 0,968 | 0,250 | 22,8° | 17,9° | 76,5% |
| 5 | 1 | 4,899 | 0,980 | 0,200 | 19,9° | 6,4° | 81,0% |
| 6 | 1 | 5,916 | 0,986 | 0,167 | 17,9° | 15,2° | 84,0% |

**Table 4b**. Numerical examples for fixed axis $l = AB = 2$

| long diameter $a$ | short diameter $b$ | focal distance $c$ | eccentricity $\varepsilon$ | latus rectum $k$ | angle $\alpha$ | angle $\beta$ | relative position $AO/AB$ |
|---|---|---|---|---|---|---|---|
| 2 | 1 | 1,732 | 0,866 | 0,500 | 32,5° | 13,1° | 36,5% |
| 3 | 1 | 2,828 | 0,943 | 0,333 | 22,8° | 12,1° | 51,1% |
| 4 | 1 | 3,873 | 0,968 | 0,250 | 18,3° | 11,3° | 60,4% |
| 5 | 1 | 4,899 | 0,980 | 0,200 | 15,6° | 10,5° | 66,8% |
| 6 | 1 | 5,916 | 0,986 | 0,167 | 13,8° | 9,9° | 71,4% |

**Table 4c**. Numerical examples for fixed axis $l = AB = 3$

| long diameter $a$ | short diameter $b$ | focal distance $c$ | eccentricity $\varepsilon$ | latus rectum $k$ | angle $\alpha$ | angle $\beta$ | relative position $AO/AB$ |
|---|---|---|---|---|---|---|---|
| 2 | 1 | 1,732 | 0,866 | 0,500 | 31,2° | 9,1° | 26,5% |
| 3 | 1 | 2,828 | 0,943 | 0,333 | 21,3° | 8,7° | 39,4% |
| 4 | 1 | 3,873 | 0,968 | 0,250 | 16,6° | 8,3° | 48,7% |
| 5 | 1 | 4,899 | 0,980 | 0,200 | 3,9° | 7,9° | 55,7% |
| 6 | 1 | 5,916 | 0,986 | 0,167 | 12,2° | 7,5° | 61,2% |

## 6.8. Conic-Section Compass or Gardeners Ellipse

When discussing the parabola, we questioned the conic-section compass as a tool. The question was whether a dedicated Euclidean parabola compass (a right angled compass) would be easier to use than the universal conic-section compass. The mathematics involved in finding the values of the required angles was a serious argument against the universal conic-section compass. For the ellipse, we can no longer use a right angled compass. Therefore, the computational task is even more complicated. When someone wants to solve an equation graphically by accurately drawing an ellipse and a parabola, and marking its intersection points, a challenging portion of arithmetic or geometry first awaits. Just as al-Bīrūnī predicted, there is no easy way. Then comes the technical implementation of using an extendable drawing arm to accurately draw the curve. al-Bīrūnī promised a nice smooth curve and stated that this curve is to be preferred to a point-by-point approach. Our attempts at producing smooth lines with the museum replicas were unsuccessful due to our lack of experience. Similarly, despite extensive practice at home, we were only able to produce consecutive single points. When drawing a parabola, we recommend using a dedicated right-angled compass, and for drawing an ellipse, we suggest using the gardener's construction shown in Figure 2.

**7. Recommendations**

This contribution describes the mathematics behind the conic-section compass. The Istanbul Museum of the History of Science and Technology in Islam exhibits a few examples. In a classroom, you might assemble one yourself. Performative methods proved to be very effective during the preparation of this workshop and those we have done in previous years. Reconstruction of an object helped to check all the details and pick out the important ones. Redoing gave insight into the performance of an object. Manipulating the rotating axis and the telescopic arm was a great challenge. During the workshop, participants worked with a replica and had to rethink the experiment. Thanks to the hands-on approach, the subject of drawing ellipses was understandable even for those with basic knowledge of mathematics. Performative methods are complementary; they cannot replace textbooks or videos. In the classroom, they stimulate public outreach. They have proven to be an enrichment to achieve pedagogical goals.

**Figure 14**: Henk Hietbrink and his home-made conic-section compass

**7. Epilogue**

The seed for this contribution was planted in 2014 when I was granted permission to access the vitrines at Prof. Dr. Fuat Sezgin's museum in Frankfurt am Main, in order to conduct experiments using the perfect compass. In his vast library, I discovered manuscripts and articles. Thanks to Prof. Dr. Fuat Sezgin's reprints, many valuable contributions have been rescued from obscurity. I was curious why the perfect compass had been included in his collection. Looking back on my ten years of work in this field, there are several noteworthy aspects. Firstly, working with a replica of the instrument was an unforgettable pleasure that I want to share with the participants of our workshops. Secondly, a graphical approach can visually demonstrate the existence of solutions. Graphically you can show that certain quadratic equations do have a solution. The ability to represent the solution's existence with two parabolas is a fantastic example. Although the solution may not be a finite fraction, it is still possible to calculate the third power root of a number numerically by stepwise adding decimals. Thirdly, the mathematics, both geometry and algebra, behind the conic-section compass is far from trivial. While it is easy to calculate the dimensions of a parabola or ellipse starting from the angles at which the device is set, the reverse path is not as clear. Woepcke proved that it can be done. Our aim is to utilize classical Greek geometric methods expressed in a modern language. Our objective is to use plain mathematics for teachers and high school students. Writing sine or cosine is short for "a ratio of one side to the other".

### Animations

A comprehensive list of sources is available at https://fransvanschooten.nl/perfectcompass.htm. Animations in GeoGebra are available at their portal: https://www.geogebra.org/m/tMxr4Mah. The Istanbul Museum of the History of Science and Technology in Islam has a website with an animated video of the perfect compass (https://www.ibtav.org/en/animations/).

### Acknowledgements

I thank the Institut für Geschichte der Arabisch-Islamischen Wissenschaften (Frankfurt am Main) for giving me permission to experiment with the exhibition models. I thank Mr. Dominique Raynaud for sharing his research findings. I thank Mrs Nihal Özdemir for inviting me to Fatih Sultan Mehmet Vakıf Üniversitesi and perform workshops with her students. I thank Yücel Aşıkoğlu for bringing in his camera obscura and his perfect compass to the workshop audience in Istanbul. I thank Hazal Yürük for her teamwork. The workshops would never have gotten off the ground, I would never have shown up at the right place, and I would never have been able to make myself understood without her efforts as a co-facilitator. The collaboration of an engineer and a mathematician in the 21st century proved that it is possible to accurately examine the insights of Arabic writing scholars from the $10^{th}$ to $12^{th}$ centuries.

### Authors

Henk Hietbrink is a mathematics teacher with an interest in the history of science and technology in Islam. He presented workshops at the International Summer School at the Prof. Dr. Fuat Sezgin Foundation for the History of Science in Islam in Istanbul, the Turkish Airlines Science Expo in Bursa, the Uludag Astrofest in Bursa, and the Science Festival in Konya. Information about his recent projects on sundials, astrolabes, muqarnas, and drawing instruments is at www.henkhietbrink.nl.

Hazal Yürük is a master student at Istanbul Technical University who wants to study differential geometry. After her Ph.D., she would like to become a researcher and university teacher. In addition to her studies, she is working as a private tutor. Henk and Hazal presented workshops together on muqarnas, sundials and astrolabes, for example a week-long course at Nesin Mathematics Village.

## Primary sources

Moḥammad ibn al-Ḥosaïn (12th century). Traité du compas parfait et de la manière de s'en server https://gallica.bnf.fr/ark:/12148/btv1b52000453w

al-Siǧzī, (10th century) Risāla fī waṣf al-Quṭūʿ al-Makhrūṭiyya http://hdl.handle.net/1887.1/item:3391451

## References

Blåsjö, V. (2021). Operationalism: An Interpretation of the Philosophy of Ancient Greek Geometry. Found Sci 27, 587–708 https://doi.org/10.1007/s10699-021-09791-4

Dupré, S. (2020) Reconstruction, Replication and Re-enactment in the Humanities and Social Sciences, Amsterdam, Amsterdam University Press

Fors, H., Principe, L., Sibum, H. (2016). From the Library to the Laboratory and Back Again: Experiment as a Tool for Historians of Science, In: Ambix, 63:2

Hendriksen, M. (2020). Rethinking Performative Methods in the History of Science In: Berichte zur Wissenschaftsgeschichte, 43-3

Hietbrink, H. (2021), What shall We Tell our Students about Ibn al-Haytham? In: The 2nd International Prof. Dr. Fuat Sezgin Symposium on History of Science in Islam Proceedings Book, Istanbul Press, pp. 227-245 https://iupress.istanbul.edu.tr/tr/book/2-uluslararasi-prof-dr-fuat-sezgin-islam-bilim-tarihi-sempozyumu-bildiriler-kitabi/chapter/what-shall-we-tell-our-students-about-ibn-al-haytham

Hogendijk, J.P. (1991). Desargues' Brouillon Project and the Conics of Apollonius: In: Centaurus volume 34, pp. 1-43 http://dx.doi.org/10.1111/j.1600-0498.1991.tb00687.x

Hogendijk, J.P. (1995). Ibn al-Haytham's Completion of the Conics. New York: In: Sources and Studies in the History of Mathematics and Physical Sciences, Springer, no. 7

Hogendijk, J.P. (1998). L'étude des sections coniques dans la tradition Arabe, In: Actes du 3ème colloque Maghrébin sur les Mathématiques Arabes, Tipaza. Alger: École Normale Supérieure, pp. 147-158

Milici, P., Plantevin, F., Salvi, M. (2020). A machine for conics and oblique trajectories, https://hal.science/hal-02530207

Raynaud, D. (2007). Le tracé continu des sections coniques à la Renaissance, In: Arabic Sciences and Philosophy , volume 17 , pp. 299 – 345, Cambridge: Cambridge University Press https://doi.org/10.1017/S0957423907000422

Rashed, R. (2003). Al-Quhi et Al-Sijzi: Sur Le Compas Parfait et le Tracé Continu des Sections Coniques, In: Arabic Sciences and Philosophy, volume 13, pp. 9-43, Cambridge: Cambridge University Press https://doi.org/10.1017/S0957423903003023

Rashed, R. (2005). Geometry and Dioptrics, In: Classical Islam, London: al-Furqan, Islamic Heritage Foundation

Rose, P.L. (1970). Renaissance Italian methods of drawing the ellipse and related curves, In: Physis, volume 12, pp 371-404, Milano: Museo Nazionale

Schooten, F. (1646), De Organica Conicarum Sectionum In Plano Descriptione, Leiden, Elzivir http://echo.mpiwg-berlin.mpg.de/MPIWG:YCF79PQK

Sezgin, F. (2001). Manuscript of Arabic Mathematical and Astronomical Treatises, Frankfurt am Main: Institut für Geschichte der Arabisch-Islamischen

Sezgin, F. (2010). Science and technology in Islam, Frankfurt am Main: Institut für Geschichte der Arabisch-Islamischen Wissenschaften https://archive.org/details/HistoryOfScienceAndTechnologyInIslam

Vittori, T. (2009). The perfect compass: conics, movement and mathematics around the 10th century, in: History and epistemology in mathematics education, Proceedings of the 6th European Summer University, pp.539-548. Vienna: Verlag Holzhausen https://hal.science/hal-00658217/document

Woepcke, F. (1874) Trois traités arabes sur le compas parfait, publiés et traduits par François Woepcke, in: Notices et extraits des manuscrits de la Bibliothèque impériale (Paris) 22/1874/1−175 (Reprinted in: F. Woepcke, Études sur les mathématiques arabo−islamiques. Nachdruck von Schriften aus den Jahren 1842−1874, Frankfurt, 1986, vol. 2, pp. 560−734 and in: Islamic Mathematics and Astronomy, vol. 66, Frankfurt am Main: Institut für Geschichte der Arabisch-Islamischen Wissenschaften, 1998, pp. 33−209